RAN-2203000205023001

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## T. Y. B. Sc. (Sem. - V) Examination March - 2023

## Mathematics : Paper - MTH-501 (New Course)

## Group Theory

**Time: 2 Hours ]** **[ Total Marks: 50**

**સૂચના : / Instructions**

(1)

**નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.**
**Fill up strictly the details of ☛ signs on your answer book**

Name of the Examination:

☛ **T. Y. B. Sc. (Sem. - V)**

Name of the Subject :

☛ **Mathematics : Paper - MTH-501 (New Course) Group Theory**

Subject Code No.: **2203000205023001**

Seat No.:

Student's Signature

(2) All questions are compulsory.
(3) Figures to the right indicate marks of corresponding question.
(4) Follow usual notations.
(5) Use of non-programmable scientific calculator is allowed.

**Q. 1. Answer the following as directed : (Any FIVE ) (10)**

(1) If $x = x^{-1}$; for every element $x$ in a group $G$, then prove that $G$ is abelian.

(2) Prove that a cyclic group is abelian.

(3) State Lagrange's Theorem for group. Justify it by giving one illustration.

(4) If $G$ is a finite group and $a \in G$, then prove that $a^{\circ(G)} = e$.

(5) Prove that every subgroup of an abelian group is normal.

(6) If $\phi : G \to G$ is an automorphism of a group $G$, then prove that $o(\phi(a)) = o(a)$; $\forall\ a \in G$.

(7) Compute $\theta^{-1}.\ \tau$ ; for the permutations $\theta = (4\ 6\ 2\ 5) \cdot (2\ 1)$ and $\tau = (1\ 3\ 5)$ in $S_6$.

(8) Which of the following permutations in $S_9$ are odd:
(a) $(1\ 5\ 3) \cdot (4\ 3\ 2\ 9) \cdot (2\ 8\ 7)$;
(b) $(9\ 1\ 7\ 4\ 2\ 3) \cdot (1\ 2\ 5\ 8\ 4) \cdot (1\ 6)$.

**Q. 2.** **Attempt any TWO.** **(10)**

(a) Define Abelian Group. Give one illustration of Non - Abelian Group. If $G$ is an abelian group, then prove that $(a \cdot b)^2 = a^2 \cdot b^2$ ; for all $a, b$ in G.

(b) Prove that a non - empty subset $H$ of a group $G$ is a subgroup of $G$ if and only if
(i) $a, b \in H \Rightarrow a \cdot b \in H$,
(ii) $a \in H \Rightarrow a^{-1} \in H$.

(c) Prove that the intersection of any two subgroups of a group is a subgroup.

**Q. 3.** **Attempt any TWO :** **(10)**

(a) Prove that the relation of *"congruence modulo H"* defines an equivalence relation on a group $G$ ; where $H$ is a subgroup of $G$.

(b) Prove that a finite group of prime order is cyclic.

(c) Let $H$ and $K$ be subgroups of a group $G$. If $HK$ is a subgroup of $G$, then prove that $HK = KH$.

**Q. 4.** **Attempt any TWO :** **(10)**

(a) If $N$, $M$ are normal subgroups of a group $G$ and $N \cap M = (e)$, then prove that $n \cdot m = m \cdot n$ ; for every element $n$ in $N$ and for every element $m$ in $M$.

(b) Define Homomorphism of Groups. Let $\phi : G \to \overline{G}$ be a homomorphism of a group G in to a group $\overline{G}$. Then prove that:
(i) $\phi(e) = \overline{e}$, where e, $\overline{e}$ are the identity elements of groups $G$ and $\overline{G}$ respectively;
(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}$; for every $x$ in $G$.

(c) Prove that the mapping $\phi : G \to G$ of a group $G$ defined by $a\phi = a^{-1}$ ; $\forall\ a \in G$ ; is an automorphism of $G$ if and only if $G$ is an abelian group.

**Q. 5.** **Attempt any TWO :** **(10)**

(a) Prove that the relation $a \equiv_\theta b$ on $S$ ; where $a, b \in S$ & $\theta \in A(S)$ is an equivalence relation on $S$.

(b) (i) Find the orbit of 5 ; for the given permutation :

$$\theta = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 5\,7\,3\,2\,1\,4\,6 \end{pmatrix}$$

(ii) Prove that there does not exist any permutation $\theta$ in $S_9$ satisfying $\theta^{-1} \cdot (7\ 2\ 9) \cdot \theta = (9\ 6\ 1\ 5)$.

(c) Given the permutations in $S_6$ ; $\sigma = (6\ 5) \cdot (3\ 1)$ and $\tau = (2\ 1) \cdot (4\ 3)$ ; find the permutation $\theta$ in $S_6$ satisfying $\theta^{-1} \cdot \sigma \cdot \theta = \tau$.